THE INTEGRABILITY OF THE DIFFERENTIAL EQUATION REPRESENTING THE SUM OF A FAMILY OF SERIES

BY

LOUIS O'SHAUGHNESSY.

A THESIS

PRESENTED TO THE FACULTY OF PHILOSOPHY OF THE UNIVERSITY OF PENNSYLVANIA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

JUNE 1911.

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NOTE.

The fundamental formulæ used in this thesis have been given me by Professor I. J. Schwatt, under whose direction the thesis has been written and to whom I am indebted for assistance rendered and for the interest he has shown.

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If we consider the series:

$$S = \frac{1}{\lfloor a_1 \rfloor a_2 \cdots \lfloor a_{p-1} \rfloor a_p} + \frac{x}{\lfloor a_1 + h_1 \rfloor a_2 + h_2 \cdots \lfloor a_p + h_p}$$
$$+ \frac{x^2}{\lfloor a_1 + 2h_1 \rfloor a_2 + 2h_2 \cdots \lfloor a_p + 2h_p} + \frac{x^3}{\lfloor a_1 + 3h_1 \rfloor a_2 + 3h_2 \cdots \lfloor a_p + 3h_p}$$
$$+ \cdots + \frac{x^k}{\lfloor a_1 + kh_1 \rfloor a_2 + kh_2 \cdots \lfloor a_p + kh_p} + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{x^n}{\prod_{r=1}^{p} \lfloor a_r + nh_r};$$

any general term

$$U_t = \frac{x^t}{\prod_{r=1}^{p} \lfloor a_r + th_r}$$

and

$$U_{t-1} = \frac{x^{t-1}}{\prod_{r=1}^{p} \lfloor a_r + (t-1)h_r}.$$

$$\therefore \frac{U_t}{U_{t-1}} = \frac{x}{\prod_{r=1}^{p} (a_r + th_r)(a_r + th_r - 1) \cdots (a_r + th_r - \overline{h_r - 1})} = \frac{x}{D}$$

and

$$\sum_{t=1}^{\infty} DU_t = x \sum_{t=1}^{\infty} U_{t-1}.$$

(This holds for $t \geq 1$.) When

$$t = 0; \; DU_t = \frac{1}{\prod_{r=1}^{p} \lfloor a_r} \; D = \frac{\prod_{r=1}^{p} (a_r)(a_r - 1) \cdots (a_r - \overline{h_r - 1})}{\prod_{r=1}^{p} \lfloor a_r}$$
$$\therefore DU_{t=0} = \frac{1}{\prod_{r=1}^{p} \lfloor a_r - h_r}.$$

And

$$\sum_{t=0}^{\infty} DU_t = \frac{1}{\prod_{r=1}^{p} \lfloor a_r - h_r} + x \sum_{t=0}^{\infty} U_t.$$

Now,

$$\begin{aligned} D &= \prod_{r=1}^{p} (th_r + a_r)(th_r + a_r - 1) \cdots (th_r + a_r - \overline{h_r - 1}) \\ &= \prod_{r=1}^{p} h_r^{h_r} \left(t + \frac{a_r}{h_r}\right)\left(t + \frac{a_r - 1}{h_r}\right) \cdots \left(t + \frac{a_r - \overline{h_r - 1}}{h_r}\right) \\ &= h_1^{h_1} h_2^{h_2} \cdots h_p^{h_p} \left(t + \frac{a_1}{h_1}\right) \cdots \left(t + \frac{a_1 - \overline{h_1 - 1}}{h_1}\right)\left(t + \frac{a_2}{h_2}\right) \\ &\qquad \cdots \left(t + \frac{a_2 - \overline{h_2 - 1}}{h_2}\right) \cdots \left(t + \frac{a_k}{h_k}\right) \cdots \left(t + \frac{a_k - \overline{h_k - 1}}{h_k}\right) \\ &\qquad \cdots \left(t + \frac{a_p}{h_p}\right) \cdots \left(t + \frac{a_p - \overline{h_p - 1}}{h_p}\right) \\ &= h_1^{h_1} \cdots h_p^{h_p} [t^{h_1 + h_2 + \cdots + h_p} + A_1 t^{h_1 + \cdots h_p - 1} + \cdots \\ &\qquad + A_k t^{h_1 + \cdots + h_p - k} + \cdots + A_{h_1 + \cdots + h_p - 1} t + A_{h_1 + \cdots + h_p}]. \end{aligned}$$

In this expression, A_1 is the sum of the second terms of the binomials in the preceding product; or in general, the sum of $(a_k - s)/h_k$ where k varies from 1 to p, and s varies from 0 to $h_k - 1$. A_2 is the sum of the products of the second terms two at a time, etc.

Since there are $\sum_{r=1}^{p} h_r$ binomials, we shall let b_{s_u} represent the general second term and let s vary from 1 to $\sum_{r=1}^{p} h_r$. From this we have:

The coefficient of $t^0 = \prod_{s=1}^{\sum_{r=1}^{p} h_r} b_s$.

The coefficient of $t = \sum_{\substack{s_1=1 \\ \cdots \\ s_{\Sigma h_r - 1}=1}}^{\Sigma h_r} \prod_{u=1}^{\Sigma h_r - 1} b_{s_u}, \quad s_1 \neq s_2 \neq \cdots \neq s_k \neq \cdots \neq s_{\Sigma h_r - 1}.$

. . . . . .

The coefficient of $t^n = \sum_{\substack{s_1=1 \\ s_2=1 \\ \cdots \\ s_{\Sigma h_r - n}=1}}^{\Sigma h_r} \prod_{u=1}^{\Sigma h_r - n} b_{s_u}, \qquad s_1 \neq s_2 \neq \cdots \neq s_{\Sigma h_r - n}.$

Now we have:

$$\sum_{t=0}^{\infty} h_1^{h_1} \cdots h_p^{h_p} \Big[t^{h_1 + \cdots + h_p} + A_1 t^{h_1 + \cdots + h_p - 1} + \cdots + A_k t^{h_1 + \cdots + h_p - k} + \cdots + A_{h_1 + \cdots + h_p - 1} t + A_{h_1 + \cdots + h_p} \Big] U_t = \frac{1}{\prod_{r=1}^{p} \lfloor a_r - h_r} + x \sum_{t=0}^{\infty} U_t.$$

However:

$$\sum_{t=0}^{\infty} t^p U_t = \sum_{v=1}^{p} \frac{\sum_{m=0}^{v-1} (-1)^m \binom{v}{m} (v-m)^p}{\underline{|v}} x^v \frac{d^v}{dx^v} S.$$

Wherein, $\sum_{t=0}^{\infty} U_t = S$. Also U_0 is finite.

Again,

$$\left(x \frac{d}{dx}\right)^p = \sum_{v=1}^{p} \frac{\sum_{m=0}^{v-1} (-1)^m \binom{v}{m} (v-m)^p}{\underline{|v}} x^v \frac{d^v}{dx^v}.$$

Therefore, we have:

$$h_1^{h_1} \cdots h_p^{h_p} \left\{ \sum_{t=0}^{\infty} t^{h_1+\cdots+h_p} U_t + A_1 \sum_{t=0}^{\infty} t^{h_1+\cdots+h_p-1} U_t \right.$$
$$+ \cdots + A_k \sum_{t=0}^{\infty} t^{h_1+\cdots+h_p-k} U_t + \cdots + A_{h_1+\ldots+h_p-1} \sum_{t=0}^{\infty} t U_t$$
$$\left. + A_{h_1+\ldots+h_p} \sum_{t=0}^{\infty} U_t \right\} = \frac{1}{\prod_{r=1}^{p} \underline{|a_r - h_r}} + xS.$$

From the above formula for $\sum_{t=0}^{\infty} t^p U_t$, we have:

$$\sum_{t=0}^{\infty} t^{h_1+\cdots+h_p} U_t = \sum_{v=1}^{h_1+\ldots+h_p} \frac{\sum_{m=0}^{v-1} (-1)^m (v-m)^{h_1+\cdots+h_p} \binom{v}{m}}{\underline{|v}} x^v \frac{d^v}{dx^v} S$$

and

$$h_1^{h_1} \cdots h_p^{h_p} \left\{ \sum_{k=0}^{h_1+\ldots+h_p} A_k \sum_{t=0}^{\infty} t^{h_1+\cdots+h_p-k} U_t \right\} = h_1^{h_1} \cdots h_p^{h_p}$$
$$\left[\sum_{k=0}^{h_1+\ldots+h_p-1} A_k \sum_{v=1}^{h_1+\ldots+h_p-k} \frac{\sum_{m=0}^{v-1} (-1)^m \binom{v}{m} (v-m)^{h_1+\cdots+h_p-k}}{\underline{|v}} x^v \frac{d^v}{dx^v} S \right.$$
$$\left. + A_{h_1+\ldots+h_p} \sum_{t=0}^{\infty} U_t \right] = \frac{1}{\prod_{r=1}^{p} \underline{|a_r - h_r}} + xS.$$

The above differential equation is evidently of order $\sum_{r=1}^{p} h_r$. Then we may write as the typical differential equation of order $\sum_{r=1}^{p} h_r$, the following:

$$h_1^{h_1}\cdots h_p^{h_p}[x^{h_1+\cdots+h_p}S^{(h_1+\cdots+h_p)}+B_1x^{h_1+\cdots+h_p-1}S^{(h_1+\cdots+h_p-1)}+\cdots$$
$$+B_kx^{h_1+\cdots+h_p-k}S^{(h_1+\cdots+h_p-k)}+\cdots+B_{h_1+\cdots+h_p-1}xS'$$
$$+B_{h_1+\cdots+h_p}S]-xS-\frac{1}{\prod_{r=1}^{p}\underline{|a_r-h_r}}=0.$$

If we consider the case in which we have but one a and one h or:

$$S=\sum_{n=0}^{\infty}\frac{x^n}{\underline{|a+nh}},$$

the differential equation is of order h and of the form:

$$h^h[x^hS^{(h)}+C_1x^{h-1}S^{(h-1)}+\cdots+C_kx^{h-k}S^{(h-k)}$$
$$+\cdots+C_{h-1}xS'+C_hS]-xS-\frac{1}{\underline{|a-h}}=0.$$

The sum

$$S=\sum_{n=0}^{\infty}\frac{x^n}{\underline{|a+nh}},$$

may be obtained by the following formula:

$$S_{a-\overline{n_1h-m}}=\frac{1}{hx^{(n_1h-m)/h}}\sum_{q=0}^{h}\frac{E^{\theta^q x^{1/h}}}{\theta^{(n_1h-m)q}}-\sum_{t=1}^{n_1-1}\frac{1}{x^t\underline{|(n_1-t)h-m}};$$

wherein for S_0, $n_1h-m=a$, also $m<h$ and

$$\theta=\cos\frac{2\pi}{h}+i\sin\frac{2\pi}{h}.$$

Let $a=3$ and $h=2$; then $n_1h-m=3$. $\therefore n_1=2$ and $m=1$. $\theta=-1$. Whence

$$S_0=\frac{1}{2x^{\frac{3}{2}}}\sum_{q=1}^{2}\frac{E^{\theta^q x^{1/2}}}{\theta^{3q}}-\sum_{t=1}^{1}\frac{1}{x^t\underline{|(2-1)2-1}}=\frac{1}{2x^{\frac{3}{2}}}\{E^{x^{\frac{1}{2}}}-E^{-x^{\frac{1}{2}}}\}-\frac{1}{x}.$$

For $x=1$,

$$S=\sum_{n=0}^{\infty}\frac{1}{\underline{|3+2n}}=\tfrac{1}{2}\{E-E^{-1}\}-1.$$

(This is easily verified.) For the case

$$S=\sum_{n=0}^{\infty}\frac{x^n}{\underline{|a_1+nh_1}\,\underline{|a_2+nh_2}},$$

we have an equation of order h_1+h_2, and of the form:

$$h_1^{h_1}h_2^{h_2}[x^{h_1+h_2}S^{(h_1+h_2)} + C_1x^{h_1+h_2-1}S^{(h_1+h_2-1)} + \cdots + C_kx^{h_1+h_2-k}S^{(h_1+h_2-k)}$$
$$+ \cdots + C_{h_1+h_2}S] - xS - \frac{1}{\lfloor a_1 - h_1 \rfloor \lfloor a_2 - h_2}} = 0.$$

The solution of this equation is the series

$$\sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 + nh_1 \rfloor \lfloor a_2 + nh_2}}.$$

For $h_1 = h_2 = 1$ we have:

$$x^2S'' + (a_1 + a_2 + 1)xS' + (a_1a_2 - x)S - \frac{1}{\lfloor a_1 - 1 \rfloor \lfloor a_2 - 1}} = 0. \quad (1)$$

Inasmuch as the a's are at our disposal the multiplicity of forms which this equation may be made to assume is evident.

For instance, if $a_1 = a_2 = 0$ [remembering that $\Pi n = n\Pi(n-1)$, $\Pi(-n)\Pi(n-1) = \pi \operatorname{cosec} n\pi$, and Πn for n a negative integer, becomes infinite. Πn being Gauss's function, which for n a positive integer equals $\lfloor n \rfloor$] we have

$$x^2S'' + xS' - xS = \frac{1}{\infty} = 0.$$

From this it is evident that if either a be zero the constant term is zero. Also, if either $a_1 < h_1$ or $a_2 < h_2$ the constant term vanishes.

The solution of $xS'' + S' - S = 0$ is

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor n \rfloor \lfloor n}}.$$

But this value of $S = J_0(2i\sqrt{x})$. (See below.) If

$$S = \sum_{n=0}^{\infty} \frac{(-x)^n}{\lfloor a_1 + nh_1 \rfloor \lfloor a_2 + nh_2}}$$

and $h_1 = h_2 = 1$, also $a_1 = 0$, the solution of $xS'' + (a_2 + 1)S' + S = 0$, is

$$S = \sum_{n=0}^{\infty} \frac{(-x)^n}{\lfloor n \rfloor \lfloor a_2 + n}},$$

or $S = x^{-a_2/2}J_{a_2}(\pm 2\sqrt{x})$. (See below.)

Now Bessel's Function

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-)^s x^{n+2s}}{2^{n+2s}\lfloor s \rfloor \lfloor n+s}} = \left(\frac{x}{2}\right)^n \sum_{s=0}^{\infty} \frac{(-)^s}{\lfloor s \rfloor \lfloor n+s}} \left(\frac{x}{2}\right)^{2s} = \left(\frac{x^2}{4}\right)^{n/2} \sum_{s=0}^{\infty} \frac{\left(-\frac{x^2}{4}\right)^s}{\lfloor s \rfloor \lfloor n+s}}$$
$$= (y)^{n/2} \sum_{s=0}^{\infty} \frac{(-y)^s}{\lfloor s \rfloor \lfloor n+s}} \left(\text{where } y = \frac{x^2}{4} \text{ and } x = \pm 2\sqrt{y}\right).$$
$$\therefore (y)^{n/2} \sum_{s=0}^{\infty} \frac{(-y)^s}{\lfloor s \rfloor \lfloor n+s}} = J_n(\pm 2\sqrt{y}).$$

In the particular case in which

$$S = \sum_{n=0}^{\infty} \frac{(-x)^n}{\underline{|n}\,\underline{|n}}, \quad S = J_0(\pm 2\sqrt{x}).$$

Then for $h_1 = h_2 = 1$, and one a zero, the solution of

$$xS'' + (a_1 + a_2 + 1)S' + S = 0,$$

may be expressed as one of the following: for $a_1 = 0$, as $x^{-a_2/2}J_{a_2}(\pm 2\sqrt{x})$; for $a_2 = 0$, as $x^{-a_1/2}J_{a_1}(\pm 2\sqrt{x})$; and for $a_1 = a_2 = 0$, as $J_0(\pm 2\sqrt{x})$. (The n being the n of Bessel's Equation: $x^2S'' + xS' + (x^2 - n^2)S = 0$).

From equation (1), page 7, it is evident that by a proper selection of the a's, we may change the coefficient of xS' to any value we may desire and yet keep the constant term zero. Thus having the solution of: $xS'' + kS' - S = 0$, ($k = a_1 + a_2 + 1$; or for $a_1 = 0$, $k = a_2 + 1$) and we have

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\underline{|n}\,\underline{|k-1+n}} = \left(\frac{-1}{x}\right)^{\frac{k-1}{2}} J_{k-1}(\pm 2i\sqrt{x}).$$

For the equation: $xS'' + kS' + S = 0$, we obtain:

$$S = \sum_{n=0}^{\infty} \frac{(-x)^n}{\underline{|n}\,\underline{|k-1+n}} \quad \text{or} \quad S = \left(\frac{1}{x}\right)^{\frac{k-1}{2}} J_{k-1}(\pm 2\sqrt{x}).$$

The only other equation of the second order arising for h_1 and h_2 both positive or only one zero, is when $h_1 = 2$, $h_2 = h_3 = \cdots = h_p = 0$. Then we have:

$$4x^2S'' + (4a_1 + 2)xS' + a_1(a_1 - 1)S - xS - \frac{1}{\underline{|a_1 - 2}\,\underline{|a_2}} = 0.$$

For a_2 negative or $a_1 < 2$, the constant term becomes zero, and we have:

$$x^2S'' + kxS' + \left(k' - \frac{x}{4}\right)S = 0 \quad \left(k = \frac{4a_1 + 2}{4};\ k' = \frac{a_1(a_1 - 1)}{4}\right)$$

and

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\underline{|a_1 + 2n}\,\underline{|a_2}} = 0,$$

(for a_2 being negative, $\Pi a_2 = \infty$); but for $a_1 < 2$ and a_2 positive, the series begins with that term in which $a_1 < 2n$.

Under the same conditions for the h's and with $a_2 \geq 0$, also $a_1 \gtrless 2$, we have the solution of the equation:

$$x^2S'' + kxS' + \left(k' - \frac{x}{4}\right)S = \frac{1}{\underline{|a_1 - 2}\,\underline{|a_2}}.$$

Whence

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\underline{|a_1 + 2n}\,\underline{|a_2}}.$$

For $a_1 < 0$,

$$S = \sum_{n \geq \left|\frac{a_1}{2}\right|}^{\infty} \frac{x^n}{\lfloor a_1 + 2n \rfloor a_2},$$

and the series does not begin until $2n \geq |a_1|$.

For $0 < a_1 < 2$, the equation is the same as for a_1 negative and the values of n range from 0 to ∞.

To return to the original equation of order $h_1 + h_2$ and of the form:

$$h_1^{h_1} h_2^{h_2} [x^{h_1+h_2} S^{(h_1+h_2)} + \cdots + C_k x^{h_1+h_2-k} S^{(h_1+h_2-k)} + \cdots + C_{h_1+h_2} S] - xS - \frac{1}{\lfloor a_1 - h_1 \rfloor a_2 - h_2} = 0,$$

wherein

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 + nh_1 \rfloor a_2 + nh_2}.$$

We may conclude that for any positive values of h_1 and h_2 there is a solution for the corresponding differential equation of order $h_1 + h_2$.

For

$$S = \sum_{n=0}^{\infty} \frac{(-x)^n}{\lfloor a_1 + nh_1 \rfloor a_2 + nh_2}$$

the differential equation is:

$$h_1^{h_1} h_2^{h_2} [x^{h_1+h_2} S^{(h_1+h_2)} + \cdots + C_{h_1+h_2-1} xS' + C_{h_1+h_2} S] + xS - \frac{1}{\lfloor a_1 - h_1 \rfloor a_2 - h_2} = 0.$$

It may be noted that this form with certain limitations previously mentioned, may be represented as a Bessel's function of the first kind.

For

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 + nh_1 \rfloor a_2 + nh_2 \rfloor a_3 + nh_3}$$

the differential equation is of the form:

$$h_1^{h_1} h_2^{h_2} h_3^{h_3} [x^{h_1+h_2+h_3} S^{(h_1+h_2+h_3)} + \cdots + C_{h_1+h_2+h_3-1} xS' + C_{h_1+h_2+h_3} S] - xS - \frac{1}{\lfloor a_1 - h_1 \rfloor a_2 - h_2 \rfloor a_3 - h_3} = 0.$$

For $h_1 = h_2 = h_3 = 1$ we have:

$$x^3 S''' + (a_1 + a_2 + a_3 + 3) x^2 S'' + (a_1 a_2 + a_2 a_3 + a_3 a_1 + a_1 + a_2 + a_3 + 1) xS' + (a_1 a_2 a_3 - x) S - \frac{1}{\lfloor a_1 - 1 \rfloor a_2 - 1 \rfloor a_3 - 1} = 0.$$

For any a zero or negative the constant term vanishes. The solution of this equation is:

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 + n \rfloor a_2 + n \rfloor a_3 + n}.$$

This value of S may be designated as $\phi_{\substack{a_1, a_2, a_3 \\ 1. \ 1. \ 1.}}(x)$.

If any a be negative the series begins with $nh \geq |a|$, for again $\Pi a = \infty$. For instance if $a_1 = -q$;

$$\frac{x^0}{\Pi(-q+0)\lfloor a_2 \rfloor a_3} = 0, \qquad \frac{x^1}{\Pi(-q+1)\lfloor a_2 + 1 \rfloor a_3 + 1} = 0,$$

etc., until $n = q$, then

$$\frac{x^q}{\Pi(0)\lfloor a_2 + q \rfloor a_3 + q} = \frac{x^q}{\lfloor a_2 + q \rfloor a_3 + q}.$$

And

$$S = \sum_{n=q}^{\infty} \frac{x^n}{\Pi(-q+n)\lfloor a_2 + n \rfloor a_3 + n}$$

or $S = \phi_{\substack{-q, a_2, a_3 \\ 1. \ 1. \ 1.}}(x)$.

(See Gray and Mathew's Bessel's Functions.)

If any h be zero (say h_1), the equation becomes

$$h_2^{h_2} h_3^{h_3} \left[x^{h_2+h_3} S^{(h_2+h_3)} + \cdots + C_{h_2+h_3} S \right] - xS - \frac{1}{\lfloor a_2 - h_2 \rfloor a_3 - h_3} = 0,$$

and

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 \rfloor a_2 + nh_2 \rfloor a_3 + nh_3}$$

or $S = \phi_{\substack{a_1, a_2, a_3 \\ 0, h_2, h_3}}(x)$. (The first term of the differential equation does not vanish, as might be inferred from h_1 being zero.) We may say

$$S = \frac{1}{\lfloor a_1} \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_2 + nh_2 \rfloor a_3 + nh_3},$$

which vanishes for a negative value of a_1. So for $h_1 = 0$ the order of the equation is lowered to $h_2 + h_3$, but the solution exists except as above stated.

$$\left[\text{Let us designate } S = \sum_{n=0}^{\infty} \frac{x^n}{\prod_{r=1}^{p} \lfloor a_r + nh_r} \text{ as } \phi_{\substack{a_1 \ldots a_p \\ h_1 \ldots h_p}}(x). \right]$$

Then for three factorials the differential equation has the usual solution; the limitations for negative a's being that the series does not begin until $nh \geq a$, wherein this a is the greatest absolutely, of all the negative a's.

The general solution of the equation of order $h_1 + h_2 + h_3$ is $\phi_{\substack{a_1, a_2, a_3 \\ h_1, h_2, h_3}}(x)$ For $a_1 = h_1 = 0$, the order becomes $h_2 + h_3$ and the solution is $\phi_{\substack{0, a_2, a_3 \\ 0, h_2, h_3}}(x)$. For a_1 negative and $h_1 = 0$, $\phi_{\substack{a_1, a_2, a_3 \\ 0, h_2, h_3}}(x) = 0$. For a_1 negative and h_1 positive, $\phi_{\substack{a_1 a_2 a_3 \\ h_1 h_2 h_3}}(x)$ is an infinite series beginning with $nh_1 \geq |a_1|$.

For

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a - nh}, \quad U_t = \frac{x^t}{\lfloor a - th}, \quad U_{t-1} = \frac{x^{t-1}}{\lfloor a - (t-1)h}.$$

$$\therefore \frac{U_t}{U_{t-1}} = x(a + h - th)(a + h - th - 1) \cdots (a + h - th - \overline{h-1})$$

and

$$\sum_{t=1}^{\infty} U_t = x \sum_{t=1}^{\infty} (a + h - th)(a + h - th - 1) \cdots (a + h - th - \overline{h-1}) U_{t-1}.$$

(For $t \geq 1$.) For $t = 0$,

$$U_t = \frac{1}{\lfloor a}.$$

Then

$$\sum_{t=0}^{\infty} U_t = \frac{1}{\lfloor a} + x \sum_{t=0}^{\infty} [a + h - (t+1)h] \times [a + h - (t+1)h - 1] \cdots [a + h - (t+1)h - \overline{h-1}] U_t.$$

Or,

$$S = \frac{1}{\lfloor a} + (-h)^h x \sum_{t=0}^{\infty} \left(t - \frac{a}{h}\right)\left(t - \frac{a-1}{h}\right) \cdots \left(t - \frac{a - \overline{h-1}}{h}\right) U_t.$$

$$S = \frac{1}{\lfloor a} + (-h)^h x \sum_{t=0}^{\infty} (t^h + A_1 t^{h-1} + \cdots + A_{h-1} t + A_h) U_t$$

$$= \frac{1}{\lfloor a} + (-h)^h x \sum_{k=0}^{h-1} A_k \sum_{v=1}^{h-k} \frac{\sum_{m=0}^{v-1} (-1)^m \binom{v}{m} (v-m)^{h-k}}{\lfloor v} x^v \frac{d^v}{dx^v} S + x \sum_{t=0}^{\infty} A_h U_t.$$

Let $A_h \equiv B_h$, and

$$S = \frac{1}{\lfloor a} + (-h)^h x \{x^h S^{(h)} + B_1 x^{(h-1)} S^{(h-1)} + \cdots + B_{h-1} x S' + B_h S\}.$$

The equation is of the form:

$$x^{h+1} S^{(h)} + B_1 x^h S^{(h-1)} + \cdots + B_k x^{h-(k-1)} S^{(h-k)} + \cdots + B_{h-1} x^2 S' + B_h x S - (-h)^{-h} S + \frac{-h^{-h}}{\lfloor a} = 0.$$

The solution is:

$$S = \sum_{n=0}^{E(a/h)} \frac{x^n}{\lfloor a - nh},$$

this being a finite series ending with the term before which nh becomes $> a$ (for $\Pi(a - nh) = \infty$, for $nh > a$).

If

$$S = \sum_{n=0}^{E\left[\frac{a}{h}\right]} \frac{(-x)^n}{\lfloor a - nh},$$

the equation is:

$$x^{h+1}S^h + B_1 x^h S^{(h-1)} + \cdots + B_h x S + (-h)^{-h} S - \frac{(-h)^{-h}}{\lfloor a} = 0,$$

and S (above) is the solution. For a negative the series does not exist; and for $a = 0$ there is but one term.

As an illustration of

$$S = \sum_{n=0}^{E(a/h)} \frac{x^n}{\lfloor a - nh};$$

let $h = 2$.

$$U_t = \frac{x^t}{\lfloor a - 2t}; \quad U_{t-1} = \frac{x^{t-1}}{\lfloor a - 2(t-1)}.$$

$$\sum_{t=1}^{\infty} U_t = x \sum_{t=1}^{\infty} (a + 2 - 2t)(a + 1 - 2t) U_{t-1}.$$

$$= (-2)^2 x \sum_{t=1}^{\infty} \left(t - \frac{a+2}{2}\right)\left(t - \frac{a+1}{2}\right) U_{t-1}.$$

For $t = 0$,

$$U_t = \frac{1}{\lfloor a}.$$

$$\therefore \sum_{t=0}^{\infty} U_t = \frac{1}{\lfloor a} + 4x \sum_{t=0}^{\infty} \left(t + 1 - \frac{a+2}{2}\right)\left(t + 1 - \frac{a+1}{2}\right) U_t.$$

$$= \frac{1}{a} + 4x \sum_{t=0}^{\infty} \left(t^2 - \frac{2a-1}{2} t + \frac{a^2 - a}{4}\right) U_t;$$

or the equation is:

$$4x^3 S'' + 4x^2 \left(\frac{3 - 2a}{2}\right) S' + 4x \frac{a^2 - a}{4} S - S + \frac{1}{\lfloor a} = 0.$$

$$\therefore S = \sum_{n=0}^{E(a/2)} \frac{x^n}{\lfloor a - 2n}.$$

For

$$S = \sum_{n=0}^{E(a/2)} \frac{(-x)^n}{\lfloor a - 2n},$$

we have:

$$4x^3S'' + 4x^2\left(\frac{3-2a}{2}\right)S' + 4x\frac{a^2-a}{4}S + S - \frac{1}{\lfloor a} = 0.$$

In particular, for $a = 10$, we obtain:

$$4x^3S'' - 34x^2S' + 90xS + S - \frac{1}{\lfloor 10} = 0;$$

and

$$S = \sum_{n=0}^{5} \frac{x^n}{\lfloor 10 - 2n} \quad \text{or} \quad \frac{1}{\lfloor 10} + \frac{x}{\lfloor 8} + \frac{x^2}{\lfloor 6} + \frac{x^3}{\lfloor 4} + \frac{x^4}{\lfloor 2} + \frac{x^5}{\lfloor 0}.$$

It is very easy to show that this series satisfies the above equation. For $h = a = 10$, the series consists of only two terms,

$$\frac{1}{\lfloor 10} + \frac{x}{\lfloor 8}.$$

In general, for $h > a$ we have no series.

We shall consider next the case in which

$$S = \sum_{n=0}^{E(a_1|h_1)} \frac{x^n}{\lfloor a_1 - nh_1 \lfloor a_2 + nh_2}.$$

$$\frac{U_t}{U_{t-1}} = \frac{x(a_1 - th_1 + h_1)(a_1 - th_1 + h_1 - 1)\cdots(a_1 - th_1 + h_1 - \overline{h_1 - 1})}{(a_2 + th_2)(a_2 + th_2 - 1)\cdots(a_2 + th_2 - \overline{h_2 - 1})}.$$

$$\sum_{t=1}^{\infty}(a_2 + th_2)\cdots(a_2 + th_2 - \overline{h_2 - 1})U_t = x\sum_{t=1}^{\infty}(a_1 - th_1 + h_1)\cdots (a_1 - th_1 + h_1 - \overline{h_1 - 1})U_{t-1}.$$

For

$$t = 0,\ (a_2 + th_2)\cdots(a_2 + th_2 - \overline{h_2 - 1})U_t = \frac{1}{\lfloor a_1 \lfloor a_2}a_2\cdots(a_2 - \overline{h_2 - 1}) = \frac{1}{\lfloor a_1 \lfloor a_2 - h_2}.$$

$$\therefore \sum_{t=0}^{\infty}(a_2 + th_2)\cdots(a_2 + th_2 - \overline{h_2 - 1})U_t = \frac{1}{\lfloor a_1 \lfloor a_2 - h_2} + x\sum_{t=0}^{\infty}[a_1 - (t+1)h_1 + h_1]\cdots[a_1 - (t+1)h_1 + h_1 - \overline{h_1 - 1}]U_t.$$

And

$$h_2^{h_2}\sum_{t=0}^{\infty}\left(t + \frac{a_2}{h_2}\right)\cdots\left(t + \frac{a_2 - h_2 - 1}{h_2}\right)U_t = \frac{1}{\lfloor a_1 \lfloor a_2 - h_2} + (-h_1)^{h_1}x\sum_{t=0}^{\infty}\left(t - \frac{a_1}{h_1}\right)\cdots\left(t - \frac{a_1 - h_1 - 1}{h_1}\right)U_t,$$

or the differential equation is of the form:

$$h_2^{h_2}\{x^{h_2}S^{(h_2)} + C_1x^{h_2-1}S^{(h_2-1)} + \cdots + C_{h_2-1}xS' + C_{h_2}S\} = \frac{1}{\lfloor a_1 \lfloor a_2 - h_2}$$

$$+ (-h_1)^{h_1}\{x^{h_1+1}S^{(h_1)} + D_1x^{h_1}S^{(h_1-1)} + \cdots + D_{h_1-1}x^2S' + D_{h_1}xS\}.$$

The solution of course, is

$$S = \sum_{n=0}^{E(a_1/h_1)} \frac{x^n}{\lfloor a_1 - nh_1 \rfloor a_2 + nh_2},$$

this being a finite series for all positive values of a_1; and terminating with

$$n = E\left(\frac{a_1}{h_1}\right).$$

(For a_1 negative, the series vanishes.)

Suppose $a_1 = 8$, $h_1 = 2$, $a_2 = 7$ and $h_2 = 3$. Then

$$S = \sum_{n=0}^{4} \frac{x^n}{\lfloor 8 - 2n \rfloor 7 + 3n}.$$

From the preceding formula we obtained an equation of order 3, or

$$3^3 \sum_{t=0}^{\infty} (t + \tfrac{7}{3})(t + \tfrac{6}{3})(t + \tfrac{5}{3})U_t = \frac{1}{\lfloor 8 \lfloor 4} + (-2)^2x \sum_{t=0}^{\infty} (t - \tfrac{8}{2})(t - \tfrac{7}{2})U_t,$$

or the form:

$$27\left\{\sum_{t=0}^{\infty} t^3U_t + 6\sum_{t=0}^{\infty} t^2U_t + \tfrac{107}{9}\sum_{t=0}^{\infty} tU_t + \tfrac{70}{9}\sum_{t=0}^{\infty} U_t\right\} = \frac{1}{\lfloor 8 \lfloor 4}$$

$$+ 4x\left\{\sum_{t=0}^{\infty} t^2U_t - \tfrac{15}{2}\sum_{t=0}^{\infty} tU_t + 14\sum_{t=0}^{\infty} U_t\right\}.$$

Now

$$\sum_{t=0}^{\infty} t^3U_t = xS' + 3x^2S'' + x^3S''',$$

$$\sum_{t=0}^{\infty} t^2U_t = xS' + x^2S'', \quad \sum_{t=0}^{\infty} tU_t = xS' \quad \text{and} \quad \sum_{t=0}^{\infty} U_t = S.$$

$$\therefore 27x^3S''' + (243x^2 - 4x^3)S'' + (510x + 26x^2)S'$$

$$+ 210S - 56xS - \frac{1}{\lfloor 8 \lfloor 4} = 0.$$

The solution,

$$S = \sum_{n=0}^{4} \frac{x^n}{\lfloor 8 - 2n \rfloor 7 + 3n},$$

may be very easily verified.

If we retain the general values of a_1 and a_2 more general forms are developed. Again it may be noticed that since the greater h controls the

order of the equation, one may vary the other h and in this manner produce a very great number of equations of order equal to the greater h.

For

$$S = \sum_{n=0}^{E(a_1/h_1)} \frac{x^n}{\lfloor a_1 - 2n \rfloor a_2 + 3n},$$

we divide U_t by U_{t-1}, collect and substitute the values of $\sum_{t=0}^{\infty} t^v U_t$ arriving at the following:

$$27x^3S''' + (54 + 27a_2 - 4x)x^2S'' + (9a_2^2 + 9a_2 + 6 + 4a_1x - 2x)xS'$$
$$+ (a_2^3 - 3a_2^2 + 2a_2 - a_1^2x - 3a_1x - 2x)S - \frac{1}{\lfloor a_1 \rfloor a_2 - 3} = 0.$$

For $a_2 = 0$, the constant term vanishes; for $a_1 = 0$, the series has but one term.

However, with $a_2 \leq 2$, or a_1 negative, the various forms are evident and we may write it as a general equation of the form:

$$x^3S''' + (k + k_1x)x^2S'' + (k_2 + k_3x)xS' + (k_4 + k_5x)S = 0.$$

In the following case and the succeeding one we shall see other forms for an equation of the third order. For

$$S = \sum_{n=0}^{E(a/h)} \frac{x^n}{\lfloor a_1 - nh_1 \rfloor a_2 - nh_2 \rfloor a_2 + nh_2};$$

$$\frac{U_t}{U_{t-1}} = x\frac{(a_1 - th_1 + h_1)\cdots}{(a_3 + th_3)\cdots}$$
$$\times \frac{(a_1 - th_1 + h_1 - \overline{h_1 - 1})(a_2 - th_2 + h_2)\cdots(a_2 - th_2 + h_2 - \overline{h_2 - 1})}{(a_3 + th_3 - \overline{h_3 - 1})}.$$

For $t = 0$, we have

$$D \cdot U_t = (a_3)(a_3 - 1)\cdots(a_3 - \overline{h_3 - 1})\frac{1}{\lfloor a_1 \rfloor a_2 \rfloor a_3} = \frac{1}{\lfloor a_1 \rfloor a_2 \rfloor a_3 - h_3}.$$

Or

$$\sum_{t=0}^{\infty}(a_3 + th_3)\cdots(a_3 + th_3 - \overline{h_3 - 1})U_t = \frac{1}{\lfloor a_1 \rfloor a_2 \rfloor a_3 - h_3}$$
$$+ x\sum_{t=0}^{\infty}(a_1 - th_1)\cdots(a_1 - th_1 - \overline{h_1 - 1})(a_2 - th_2)\cdots(a_2 - th_2 - \overline{h_2 - 1})U_t.$$

And

$$h_3^{h_3}\sum_{t=0}^{\infty}\left(t + \frac{a_3}{h_3}\right)\cdots\left(t + \frac{a_3 - \overline{h_3 - 1}}{h_3}\right)U_t = \frac{1}{\lfloor a_1 \rfloor a_2 \rfloor a_3 - h_3}$$
$$+ (-h_1)^{h_1}(-h_2)^{h_2}\sum_{t=0}^{\infty}\left(t - \frac{a_1}{h_1}\right)\cdots\left(t - \frac{a_1 - \overline{h_1 - 1}}{h_1}\right)$$
$$\times\left(t - \frac{a_2}{h_2}\right)\cdots\left(t - \frac{a_2 - \overline{h_2 - 1}}{h_2}\right)U_t.$$

The differential equation is of the form:

$$h_3^{h_3}[x^{h_3}S^{(h_3)} + C_1x^{h_3-1}S^{(h_3-1)} + \cdots + C_kx^{h_3-k}S^{(h_3-k)} + \cdots + C_{h_3}S]$$

$$= \frac{1}{\lfloor a_1 \rfloor \lfloor a_2 \rfloor \lfloor a_3 - h_3}} + (-h_1)^{h_1}(-h_2)^{h_2}[x^{h_1+h_2+1}S^{(h_1+h_2)}$$

$$+ \cdots + D_kx^{h_1+h_2-k+1}S^{(h_1+h_2-k)} + \cdots + D_{h_1+h_2}xS].$$

The solution is $\phi_{\substack{a_1, & a_2, & a_3 \\ -h_1, & -h_2, & h_3}}(x)$, a finite series terminating with n equal to the less of the two greatest integers in a_1/h_1 and a_2/h_2.

To illustrate: Let $h_1 = 1$, $h_2 = 1$, $h_3 = 1$, $a_1 = 5$, $a_2 = 6$ and $a_3 = 9$. Then $E(a_1/h_1) = 5$, and $E(a_2/h_2) = 6$. Whence

$$S = \sum_{n=0}^{5} \frac{x^5}{\lfloor 5 - n \rfloor \lfloor 6 - n \rfloor \lfloor 9 + n}} \equiv \phi_{\substack{5, & 6, & 9 \\ -1, & -1, & 1}}(x).$$

The equation is as follows:

$$x^3S'' - (x + 10x^2)S + (30x - 9)S + \frac{1}{\lfloor 5 \rfloor \lfloor 6 \rfloor \lfloor 8}} = 0,$$

or

$$S = \frac{1}{\lfloor 5 \rfloor \lfloor 6 \rfloor \lfloor 9}} + \frac{x}{\lfloor 4 \rfloor \lfloor 5 \rfloor \lfloor 10}} + \frac{x^2}{\lfloor 3 \rfloor \lfloor 4 \rfloor \lfloor 11}} + \frac{x^3}{\lfloor 2 \rfloor \lfloor 3 \rfloor \lfloor 12}}$$

$$+ \frac{x^4}{\lfloor 1 \rfloor \lfloor 2 \rfloor \lfloor 13}} + \frac{x^5}{\lfloor 0 \rfloor \lfloor 1 \rfloor \lfloor 14}}.$$

If we use

$$S = \sum_{n=0}^{E(a/h)} \frac{x^n}{\lfloor a_1 - n \rfloor \lfloor a_2 - n \rfloor \lfloor a_3 + n}};$$

we obtain

$$\sum_{t=0}^{\infty}(t + a_3)U_t = \frac{1}{\lfloor a_1 \rfloor \lfloor a_2 \rfloor \lfloor a_3 - 1}} + x\sum_{t=0}^{\infty}(t - a_1)(t - a_2)U_t$$

or

$$x^3S'' - [x + (a_1 + a_2 - 1)x^2]S' + (a_1a_2x - a_3)S + \frac{1}{\lfloor a_1 \rfloor \lfloor a_2 \rfloor \lfloor a_3 - 1}} = 0.$$

This may be written

$$x^3S'' - [1 + kx]xS' + (k_1x - k_2)S + \frac{1}{\lfloor a_1 \rfloor \lfloor a_2 \rfloor \lfloor a_3 - 1}} = 0.$$

Now since k_2 (or a_3) is at our disposal and may be made zero, we have: $x^2S'' - [kx + 1]S' + k_1S = 0$, and $S = \phi_{\substack{a_1, & a_2, & 0. \\ -1. & -1. & 1.}}(x)$ for $k = a_1 + a_2 - 1$ and $k_1 = a_1a_2$. For instance, to solve $x^2S'' - (6x + 1)S' + 12S = 0$, we have $a_1a_2 = 12$, $a_1 + a_2 - 1 = 6$ or $a_1 = 3$ or 4 and $a_2 = 4$ or 3, whence $S = \phi_{\substack{3. & 4. & 0. \\ -1. & -1. & 1.}}(x)$. For the cases in which the a's are fractional see the end of this discussion.

For $a_3 < 0$ the constant term vanishes and $S = \phi_{\substack{a_1, a_2, a_3 \\ -1, -1, 1}}(x)$. This series is finite, beginning with $nh_3 \geq |a_3|$ and ending with the smaller of $E(a_1/h_1)$ and $E(a_2/h_2)$. We may conclude that the order is equal to the sum of those h's (either positive or negative) which give the greater sum; that the solution is a finite series terminating with $n = E(a_k/h_k)$, where we take the smallest integer for any value of k, and in which h_k is one of the negative h's.

It hardly need be added that for three factorials with all h's positive the solution is always possible and the series is an infinite one.

For the general case we have

$$S = \sum_{n=0}^{E(a_m/h_m)} \frac{x^n}{\prod_{m=1}^{r} \lfloor a_m + nh_m \prod_{m=r+1}^{p} \lfloor a_m - nh_m}.$$

In $E\left(\frac{a_m}{h_m}\right)$, the least value is taken for $r < m \leq p$.

$$U_t = \frac{x^t}{\prod_{m=1}^{r} \lfloor a_m + th_m \prod_{m=r+1}^{p} \lfloor a_m - th_m},$$

$$U_{t-1} = \frac{x^{t-1}}{\prod_{m=1}^{r} \lfloor a_m + (t-1)h_m \prod_{m=r+1}^{p} \lfloor a_m - (t-1)h_m},$$

$$\frac{U_t}{U_{t-1}} = x \frac{\prod_{m=r+1}^{p} (a_m - th_m + h_m) \cdots (a_m - th_m + h_m - \overline{h_m - 1})}{\prod_{m=1}^{r} (a_m + th_m) \cdots (a_m + th_m - \overline{h_m - 1})}$$

and

$$\sum_{t=1}^{\infty} \prod_{m=1}^{r} (a_m + th_m)(a_m + th_m - 1) \cdots (a_m + th_m - \overline{h_m - 1}) U_t$$
$$= x \sum_{t=1}^{\infty} \prod_{m=r+1}^{p} (a_m - th_m + h_m)(a_m + th_m + h_m - 1)$$
$$\cdots (a_m - th_m + h_m - \overline{h_m - 1}) U_{t-1}.$$

For $t = 0$ we have:

$$\prod_{m=1}^{r} a_m(a_m - 1) \cdots (a_m - \overline{h_m - 1}) \frac{1}{\prod_{m=1}^{r} \lfloor a_m \prod_{m=r+1}^{p} \lfloor a_m} = \frac{1}{\prod_{m=1}^{r} \lfloor a_m - h_m \prod_{m=r+1}^{p} \lfloor a_m},$$

and we obtain:

$$\sum_{t=0}^{\infty} \prod_{m=1}^{r} (a_m + th_m) \cdots (a_m + th_m - \overline{h_m - 1}) U_t = \frac{1}{\prod_{m=1}^{r} \lfloor a_m - h_m \prod_{m=r+1}^{p} \lfloor a_m}$$
$$+ x \sum_{t=0}^{\infty} \prod_{m=r+1}^{p} [a_m - (t+1)h_m + h_m] \cdots [a_m - (t+1)h_m + h_m - \overline{h_m - 1}] U_t.$$

$$\therefore \left[\prod_{m=1}^{r} h_m^{h_m}\right]\sum_{t=0}^{\infty}\left(t_1^{\sum h_m}+A_1 t^{\sum h_m-1}+\cdots+A_k t^{\sum h_m-k}+\cdots+A_{\sum h_m-1}t+A_{\sum h_m}\right)U_t$$

$$=\frac{1}{\prod_{m=1}^{r}\lfloor a_m-h_m \prod_{m=r+1}^{p}\lfloor a_m}+x\left[\prod_{m=r+1}^{p}(-h_m)^{h_m}\right]\sum_{t=0}^{\infty}(t_{r+1}^{\sum h_m}+B_1 t^{\sum h_m-1}+\cdots$$

$$+B_k t^{\sum h_m-k}+\cdots+B_{\sum h_m-1}t+B_{\sum h_m})U_t,$$

giving the differential equation of the form :

$$\left[\prod_{m=1}^{r} h_m^{h_m}\right]\{x_1^{\sum h_m}S^{(\sum h_m)}+C_1 x^{\sum h_m-1}S^{(\sum h_m-1)}+\cdots+C_k x^{\sum h_m-k}S^{(\sum h_m-k)}$$

$$+\cdots C_{\sum h_m-1}xS'+C_{\sum h_m}S\}=\frac{1}{\prod_{m=1}^{r}\lfloor a_m-h_m \prod_{m=r+1}^{p}\lfloor a_m}+\left[\prod_{m=r+1}^{p}(-h_m)^{hm}\right]$$

$$\{x_{r+1}^{\sum h_m+1}S^{(\sum h_m)}+\cdots+D_k x^{\sum h_m+1-k}S^{(\sum h_m-k)}+\cdots+D_{\sum h_m-1}x^2S'+D_{\sum h_m}xS\}.$$

This equation being of order equal to the greater of the two sums

$$\sum_{m=1}^{r} h_m \quad \text{and} \quad \sum_{m=r+1}^{p} h_m.$$

The solution is the S above. The series is finite, the greatest value of n being the least of the greatest integers in a_k/h_k, where $r<k\leq p$.

For the case in which we have $(-x)^n$ in the numerator of the summation the only change in the equation is that the last term of the equation is multiplied by $(-x)$ instead of by x.

Then we may say that for all integral values of h, both positive and negative, the resulting differential equation is integrable in the corresponding series.

The order is $\sum_{s=1}^{p} h_s$ for positive h's, and as given above when part positive and part negative (or subtractive).

In all cases the constant term vanishes for one negative a, and in those cases in which the h's are positive the constant term vanishes if any a should be less than the corresponding h. In cases with a negative a the series is infinite, but does not begin until $nh\geq|a|$. In cases with a negative (or subtractive) h the series is finite and ends with the term before the term in which nh becomes greater than the corresponding a. For a and h both negative no series exists.

It may be well to notice that in the above equation of order $\sum_{m=1}^{r} h_m$ or $\sum_{m=r+1}^{p} h_m$, the highest power of x in the solution being the least of the

greatest integers in a_k/h_k $(r < k \leq p)$ that any derivative of greater order than this integer plus one, does not exist. This fact in connection with the freedom of the choice of a's and h's leads to an interesting result in that we may construct our equation of order twenty-five say, and in one case have an infinite series for a solution, while by a slight change of constants we may have a finite series or no solution at all. This result is interesting only in so far as it evidences the fact that most any equation of any order may be solved by a proper selection of the a's and h's, this selection being the problem in the solution, provided of course, the equation be of a form similar to one of the general equations derived above.

Because of the fact that Gauss's Function Πn, is intelligible and finite for all values of n except when n is a negative integer (when it is infinite), we are enabled to consider cases in which the a's are fractional; $\Pi n = n\Pi(n-1)$ being the only relation necessary in order to perform the division of U_t by U_{t-1}, and the remainder of the work will result in a differential equation of an order agreeing with the conclusions heretofore stated.

For instance, if we consider

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a_1 + nh_1 \rfloor \lfloor a_2 + nh_2}$$

and let $h_1 = h_2 = 1$, we obtain:

$$x^2 S'' + (a_1 + a_2 + 1)xS' + (a_1 a_2 - x)S - \frac{1}{\lfloor a_1 - 1 \rfloor \lfloor a_2 - 1} = 0.$$

If a_1 be fractional (say $a_1 = \frac{3}{2}$) and $a_2 = 0$, we have

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\Pi(\frac{3}{2}+n)\Pi n}, \quad \text{and} \quad \frac{1}{\lfloor a_1 - 1 \rfloor \lfloor a_2} = \frac{1}{\Pi(\frac{1}{2})} = \frac{2}{\sqrt{\pi}};$$

or

$$S = \left(\frac{-1}{x}\right)^{3/4} J_{\frac{3}{2}}(\pm 2i\sqrt{x}).$$

Thus we have a solution for a fractional a, by the use of Gauss's Function.

For a a negative fraction the constant term vanishes and the remainder of the solution is as indicated above.

For the case in which h is a fraction, let us take

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\lfloor a + nh} \quad \text{where} \quad h = \frac{c}{d}.$$

Then

$$U_t = \frac{x^t}{\lfloor a + t\frac{c}{d}}, \quad U_{t-1} = \frac{x^{t-1}}{\lfloor a + (t-1)\frac{c}{d}}$$

and

$$\frac{U_t}{U_{t-1}} = x\frac{\left\lfloor a+t\frac{c}{d}-\frac{c}{d}\right.}{\left\lfloor a+t\frac{c}{d}\right.} \text{ (or better)} = x\frac{\Pi\left(a+t\frac{c}{d}-\frac{c}{d}\right)}{\Pi\left(a+t\frac{c}{d}\right)}$$

$$= x\frac{\Pi\left(a+t\frac{c}{d}-\frac{c}{d}\right)}{\left(a+t\frac{c}{d}\right)\Pi\left(a+t\frac{c}{d}-1\right)},$$

and our method fails.

However, if we define $\rho_b(a/b)$ as equivalent to

$$\frac{a}{b}\cdot\frac{a-1}{b}\cdot\frac{a-2}{b}\cdots\frac{1}{b} = \frac{\lfloor a}{b^a},$$

we are able to consider the case for h a fraction. Also

$$\rho_b\left(-\frac{a}{b}\right) \equiv \frac{a}{-b}\cdot\frac{a-1}{-b}\cdots\frac{1}{-b} = \frac{\lfloor a}{(-b)^a}.$$

With this definition of $\rho_b(a/b)$ and with a/b in this case, a fraction in its lowest terms, let us consider the case in which $h = \frac{3}{2}$ and

$$S = \sum_{n=0}^{\infty}\frac{x^n}{\rho_2(a+\frac{3}{2}n)}, \quad U_t = \frac{x^t}{\rho_2(a+\frac{3}{2}t)}, \quad U_{t-1} = \frac{x^{t-1}}{\rho_2[a+\frac{3}{2}(t-1)]}.$$

$$\therefore \frac{U_t}{U_{t-1}} = x\frac{\rho_2\left(\frac{2a+3t-3}{2}\right)}{\rho_2\left(\frac{2a+3t}{2}\right)} = \frac{x}{\frac{2a+3t}{2}\cdot\frac{2a+3t-1}{2}\cdot\frac{2a+3t-2}{2}},$$

and $(2a+3t)(2a+3t-1)(2a+3t-2)U_t = 8xU_{t-1}$, for $t \geq 1$.

For $t = 0$ we have:

$$2a(2a-1)(2a-2)\frac{1}{\rho_2(a)}.$$

Then,

$$\sum_{t=0}^{\infty}(2a+3t)(2a+3t-1)(2a+3t-2)U_t = \frac{2a(2a-1)(2a-2)}{\rho_2(a)} + 8x\sum_{t=0}^{\infty}U_t.$$

From the above we obtain a differential equation of the third order as follows:

$$27x^3S''' + (54+54a)x^2S'' + (36a^2+18a+6)xS'$$

$$+ (8a^3-12a^2+4a-8x)S - \frac{2a(2a-1)(2a-2)}{\rho_2(a)} = 0$$

For $a = 1$, we have:

$27x^3S''' + 108x^2S'' + 60xS' - 8xS = 0$, since $\dfrac{2a(2a-1)(2a-2)}{\rho_2(a)} = 0$,

and therefore,

$$S = \sum_{n=0}^{\infty} \frac{x^n}{\rho_2(1 + \frac{3}{2}n)}.$$

Let us consider the case for h negative (or subtractive).

Let

$$S = \sum_{n=0}^{E(5/\frac{3}{2})} \frac{x^n}{\rho_2(5 - \frac{3}{2}n)}, \quad U_t = \frac{x^t}{\rho_2(5 - \frac{3}{2}t)},$$

$$U_{t-1} = \frac{x^{t-1}}{\rho_2[5 - \frac{3}{2}(t-1)]} = \frac{x^{t-1}}{\rho_2(\frac{13}{2} - \frac{3}{2}t)}.$$

$$\frac{U_t}{U_{t-1}} = x \cdot \frac{\rho_2(\frac{13}{2} - \frac{3}{2}t)}{\rho_2(\frac{10}{2} - \frac{3}{2}t)} = x \cdot \frac{(13-3t)(12-3t)(11-3t)}{2^3},$$

or $8U_t = x(13-3t)(12-3t)(11-3t)U_{t-1}$, for $t \geq 1$. For $t = 0$,

$$8U_0 = 8 \cdot \frac{1}{\rho_2(5)}.$$

$$\therefore 8\sum_{t=0}^{\infty} U_t = \frac{8}{\rho_2(5)} + x\sum_{t=0}^{\infty}(10-3t)(9-3t)(8-3t)U_t.$$

or

$$27x^4S''' - 162x^3S'' + 510x^2S' - 720xS + 8S - \frac{8}{\rho_2(5)} = 0.$$

It is evident that $n \not> E\left(\dfrac{5}{\frac{3}{2}}\right)$ or 3.

Then

$$S = \frac{1}{\rho_2(5)} + \frac{x}{\rho_2(\frac{7}{2})} + \frac{x^2}{\rho_2(\frac{4}{2})} + \frac{x^3}{\rho_2(\frac{1}{2})}.$$

If we obtain S', S'' and S''' and substitute them in the above, the solution is verified.

[It is well to notice that $\rho_2(\frac{7}{2}) = \frac{7}{2} \cdot \frac{6}{2} \cdot \frac{5}{2} \cdot \rho_2(\frac{4}{2})$.]

It follows from the definition of $\rho_b(a/b)$ that

$$\rho_d\left(\frac{c}{d}\right) = \frac{\lfloor c}{\overline{d}^c}.$$

If

$$S = \sum_{n=0}^{E(a_2/h_2)} \frac{x^n}{\rho_m(a_1 + nh_1)\rho_k(a - nh_2)},$$

where $h_1 = l/m$ and $h_2 = h/k$, then

$$U_t = \frac{x^t}{\rho_m\left(a_1 + t\frac{l}{m}\right)\rho_k\left(a_2 - t\frac{h}{k}\right)},$$

and

$$U_{t-1} = \frac{x^{t-1}}{\rho_m\left[a_1 + (t-1)\frac{l}{m}\right]\rho_k\left[a_2 - (t-1)\frac{h}{k}\right]}.$$

$$\frac{U_t}{U_{t-1}} = x\frac{\rho_m\left(\frac{a_1m + tl - l}{m}\right)\rho_k\left(\frac{a_2k - th + h}{k}\right)}{\rho_m\left(\frac{a_1m + tl}{m}\right)\rho_k\left(\frac{a_2k - th}{k}\right)}$$

$$= x\frac{\frac{(a_2k - th + h)}{k}\ \frac{a_2k - th + h - 1}{k}\cdots\frac{(a_2k - th + h - \overline{h-1})}{k}}{\frac{(a_1m + tl)}{m}\frac{(a_1m + tl - 1)}{m}\cdots\frac{(a_1m + tl - \overline{l-1})}{m}}.$$

$$\therefore \frac{a_1m + tl}{m}\ \frac{a_1m + tl - 1}{m}\cdots\frac{a_1m + tl - \overline{l-1}}{m}U_t = x\frac{a_2k - th + h}{k}$$

$$\times\frac{a_2k - th + h - 1}{k}\cdots\frac{a_2k - th + h - \overline{h-1}}{k}U_{t-1},$$

for $t \geq 1$. For $t = 0$ we have:

$$\frac{a_1m}{m}\ \frac{a_1m - 1}{m}\cdots\frac{a_1m - \overline{l-1}}{m}\frac{1}{\rho_m(a_1)\rho_k(a_2)}$$

or

$$\frac{1}{\rho_m\left(\frac{a_1m - l}{m}\right)\rho_k(a_2)}.$$

$$\therefore \sum_{t=0}^{\infty}\frac{a_1m + tl}{m}\ \frac{a_1m + tl - 1}{m}\cdots\frac{a_1m + tl - \overline{l-1}}{m}U_t = \frac{1}{\rho_m\left(a_1 - \frac{l}{m}\right)\rho_k(a_2)}$$

$$+ x\sum_{t=0}^{\infty}\frac{a_2k - th}{k}\ \frac{a_2k - th - 1}{k}\cdots\frac{a_2k - th - \overline{h-1}}{k}U_t.$$

Then

$$\left(\frac{l}{m}\right)^l\sum_{t=0}^{\infty}\left(t + \frac{a_1m}{l}\right)\left(t + \frac{a_1m - 1}{l}\right)\cdots\left(t + \frac{a_1m - l - 1}{l}\right)U_t$$

$$= \frac{1}{\rho_m\left(a_1 - \frac{l}{m}\right)\rho_k(a_2)} + \left(-\frac{h}{k}\right)^h x\sum_{t=0}^{\infty}\left(t - \frac{a_2k}{h}\right)\left(t - \frac{a_2k - 1}{h}\right)$$

$$\cdots\left(t - \frac{a_2k - \overline{h-1}}{h}\right)U_t.$$

This leads to a differential equation of order equal to the greater of h or l, and the form is:

$$\left(\frac{l}{m}\right)^l \{x^l S^{(l)} + A_1 x^{l-1} S^{(l-1)} + \cdots + A_r x^{l-r} S^{(l-r)} + \cdots + A_{l-1} x S' + A S\}$$

$$= \frac{1}{\rho_m\left(a_1 - \frac{l}{m}\right)\rho_k(a_2)} + \left(-\frac{h}{k}\right)^h \{x^{h+1} S^{(h)} + B_1 x^h S^{(h-1)}$$

$$+ \cdots + B_r x^{h-r+1} S^{(h-r)} + \cdots + B_{h-1} x^2 S' + B_h x S\}.$$

The solution is:

$$S = \sum_{n=0}^{E\left[a_2 / \frac{h}{k}\right]} \frac{x^n}{\rho_m\left(a_1 + n\frac{l}{m}\right)\rho_k\left(a_2 - n\frac{h}{k}\right)}$$

or

$$\phi_{\substack{a_1, a_2, \\ \frac{l}{m}, -\frac{h}{k}}}(x).$$

The result of the use of $\rho_b(a/b)$ is that we do obtain some results for fractional h's, while with the Π function, as well as with the ordinary factorial, no results are obtainable.

For the fractional h's the order of the differential equation always depends upon the numerators of the h's, whereas for the integral values of h the order is a similar function of the h's themselves.

Thus by the use of the ordinary factorial, the Π function and the ρ function, we have been able to arrive at definite results for all values of the a's and the h's except when both are negative. In the latter case too, the result may be said to be definite for negative integral values, for then the Π function is infinite and the series does not exist.